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# Finding Solitons in Bifurcations of Stationary Solutions of Complex Ginzburg–Landau Equation

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Nonlinear dissipative systems, particularly optical dissipative solitons are well described by complex Ginzburg–Landau equation. Solutions of two- and three-dimensional complex cubic-quintic Ginzburg–Landau equation assuming exponential dependence on propagation parameter are studied. Approximate analytical stationary solutions of cubic-quintic Ginzburg–Landau equation are found by solving systems of ordinary differential equations. We are solving two-point boundary problems using adapted shooting method. Stable and unstable branches of the bifurcation diagram are identified using linear stability analysis. In this way we established conditions for generation and propagation of stable dissipative solitons in two and three dimensions. These results are in agreement with numerical simulation of cubic-quintic Ginzburg–Landau equation and the recently established approach based on variational method generalized to dissipative systems and therein established stability criterion.

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## 1. Introduction

There is a growing interest for optical solitons as form preserving self-trapped structures. Spatial and spatiotemporal solitons are good candidates in all-optical signal processing since they are self-guided in bulk media [1]. Stable operation of laser systems, closely related to the issue of dissipative soliton stability, is crucial for generating ultrashort pulses [2]. In order to generate few-parameters family of solitons with either two or three transverse dimension the diffraction and/or dispersion have to be compensated by spatial and/or temporal self-focusing [3]. However, real systems are generally dissipative, thus, linear and nonlinear gain

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and loss have to be taken into account. Nonlinear dissipative systems, particularly optical dissipative solitons are well described by complex Ginzburg–Landau equation [4]. Dispersion and diffraction in optical pulse are compensated by interplay of opposite sign cubic and quintic nonlinearities of the complex cubic-quintic Ginzburg–Landau equation (CQGLE):

$$\mathrm{i}\frac{\partial E}{\partial z} + \Delta E + |E|^2 E + \nu |E|^4 E = \mathrm{i}\delta E + \mathrm{i}\varepsilon |E|^2 E + \mathrm{i}\mu |E|^4 E + \mathrm{i}\beta \Delta E, \tag{1}$$

where $E$ is the normalized complex envelope of the optical field. $\Delta E$ stand for the $D$-dimensional Laplacian describing beam diffraction and/or anomalous group velocity dispersion, where transverse dimension $D = 1$, 2, 3 corresponds to space ($x$ and $y$) and time ($t$) variables. In order to prevent the wave collapse the saturating nonlinearity is required. Therefore, cubic and quintic nonlinearities have to have opposite signs, i.e., parameter $\nu$ is negative. Depending on the sign of the dissipative parameter $\delta$ the first term is either linear gain or loss. The cubic and quintic gain-loss terms contain respectively parameters $\varepsilon$ and $\mu$. The last term accounts for the parabolic gain if $\beta > 0$. A prerequisite for generation of dissipative solitons is a simultaneous balance of not only diffraction and/or dispersion with self-focusing but also gain with loss reducing, for a given set of parameters, a family of solutions to a fixed solution. One has to resort to computer simulations in order to investigate the solutions of such an equation. General dynamical properties of Eq. (1) are rather complex, making analytical approximation highly desirable. In the recently established approach we investigated multi-dimensional CQGLE using variational method generalized to dissipative systems and therein established stability criterion [5, 6].

Here, we study approximative solutions of two- and three-dimensional CQGLE assuming exponential dependence on propagation parameter $E(r, z) = E_0 \exp[\mathrm{i}\Omega z]$. This propagation parameter $\Omega$ corresponds to the spatial frequency. The soliton radius is supposed to be $r = \sqrt{x^2 + y^2 + t^2}$ imposing a constraint to independent transverse variables $x$, $y$, and $t$. As a consequence, the input pulses must be symmetric with respect to those variables [3, 6]. In this case the approximative analytical stationary solutions of CQGLE can be found by solving systems of ordinary differential equations. We are solving two-point boundary problems using an adapted shooting method. Stable and unstable branches of the bifurcation diagram are identified. In this way we established conditions for generation and propagation of stable dissipative solitons in two and three dimensions. These results are in agreement with numerical simulation of CQGLE and the recently established approach based on variational method [5, 6].

## 2. Eigenvalue problem

In order to study dynamics of dissipative solitons described by a $(D + 1)$--dimensional CQGLE we express the field $E$ through the amplitude $A$ and the phase $\Psi$, both real functions of $z$:

$$E = A \exp[\mathrm{i}\,\Psi]. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and separating the real and imaginary parts, we get two equations

$$\frac{1}{2}\frac{\partial A^2}{\partial z} + \nabla(A^2\nabla\Psi) - \delta A^2 - \varepsilon A^4 - \mu A^6 - \beta[A\Delta A - A^2(\nabla\Psi)^2] = 0, \tag{3}$$

and

$$A\Delta A - A^2\frac{\partial\Psi}{\partial z} - A^2(\nabla\Psi)^2 + (1 + \nabla\Psi A^2)A^4 + \beta\nabla(A^2\nabla\Psi) = 0. \tag{4}$$

Since phase appears in Eqs. (3), (4) only through its derivatives, new variables $\boldsymbol{C} = A^2\nabla\Psi$ and $\partial\Psi/\delta z = \Omega$ can be introduced. The spatial frequency $\Omega$ is the eigenvalue we are looking for. After some algebra we get

$$\Delta A = \frac{C^2}{A^3} + \frac{\Omega - \beta\delta}{1+\beta^2}A - \frac{1+\beta\varepsilon}{1+\beta^2}A^3 - \frac{\nu + \beta\mu}{1+\beta^2}A^5, \tag{5}$$

$$\nabla\boldsymbol{C} = \frac{\delta + \beta\Omega}{1+\beta^2}A^2 + \frac{\varepsilon - \beta}{1+\beta^2}A^4 + \frac{\mu - \beta\nu}{1+\beta^2}A^6. \tag{6}$$

For radial symmetry $\boldsymbol{C} = C\boldsymbol{e}_r$ as well as

$$\nabla\boldsymbol{C} = \nabla_r C + \frac{D-1}{r}C \tag{7}$$

and

$$\Delta A = \frac{1}{r^{D-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{D-1}\frac{\mathrm{d}A}{\mathrm{d}r}\right). \tag{8}$$

In this case Eqs. (5), (6) can be written as a system of three ordinary differential equations (ODE):

$$\frac{\mathrm{d}A}{\mathrm{d}r} = B, \tag{9}$$

$$\frac{\mathrm{d}B}{\mathrm{d}r} = -\frac{D-1}{r}B + \frac{C^2}{A^3} - \alpha_1 A - \alpha_3 A^3 - \alpha_5 A^5, \tag{10}$$

and

$$\frac{\mathrm{d}C}{\mathrm{d}r} = -\frac{D-1}{r}C + \alpha_2 A^2 + \alpha_4 A^4 + \alpha_6 A^6, \tag{11}$$

where

$$\alpha_1 = -\frac{\Omega - \beta\delta}{1+\beta^2}, \quad \alpha_3 = \frac{1+\beta\varepsilon}{1+\beta^2}, \quad \alpha_5 = \frac{\nu+\beta\mu}{1+\beta^2},$$
$$\alpha_2 = \frac{\delta+\beta\Omega}{1+\beta^2}, \quad \alpha_4 = \frac{\varepsilon-\beta}{1+\beta^2}, \quad \alpha_6 = \frac{\mu-\beta\nu}{1+\beta^2}. \tag{12}$$

For practical numerical resolution it is convenient to introduce substitutions $B = FA$ and $C = GA^2$. As a consequence, the system of Eqs. (9)–(11) reduces into

$$\frac{\mathrm{d}A}{\mathrm{d}r} = FA, \tag{13}$$

$$\frac{\mathrm{d}F}{\mathrm{d}r} = -\frac{D-1}{r}F - F^2 + G^2 - \alpha_1 - \alpha_3 A^2 - \alpha_5 A^4, \tag{14}$$

and

$$\frac{\mathrm{d}G}{\mathrm{d}r} = -\frac{D-1}{r}G - 2GF + \alpha_2 + \alpha_4 A^2 + \alpha_6 A^4. \tag{15}$$

For $r = 0$ functions $F$ and $G$ are zero and $A = A_0$. For $r \to \infty$ the function $A$ asymptotically tends to zero. Numerical solutions of Eqs. (13)–(15) obtained by shooting method are presented in Figs. 1–4.

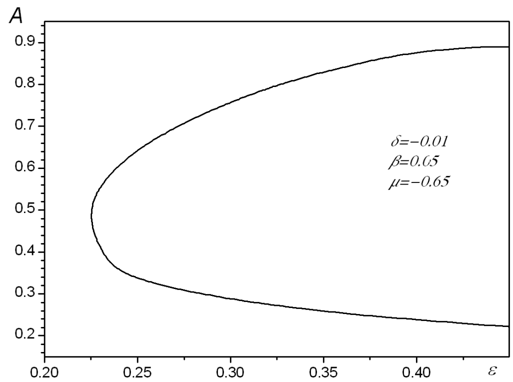


Fig. 1. Upper stable and lower unstable bifurcation curve of amplitude $A$ as functions of the parameter $\varepsilon$ for $\mu = -0.65$ in 2D case.

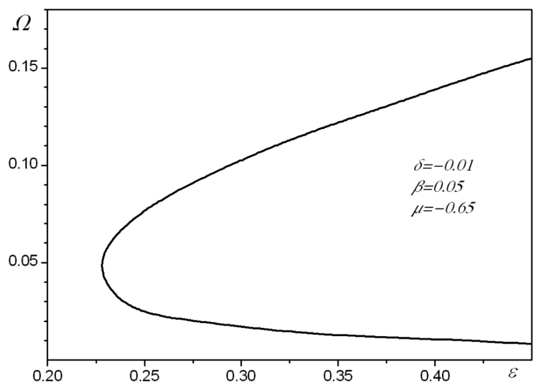


Fig. 2. Bifurcation curve of spatial frequency $\Omega$ as functions of the parameter $\varepsilon$ in 2D case.

Indeed, the soliton is a localized structure thus its amplitude has to vanish exponentially out of localization region whenever the spatial frequency $\Omega$ becomes

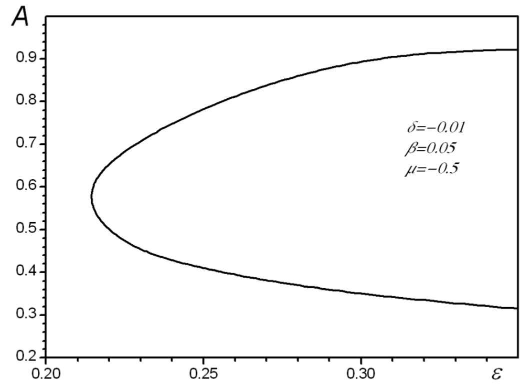


Fig. 3. Upper stable and lower unstable bifurcation curve of amplitude $A$ as functions of the parameter $\varepsilon$ for $\mu = -0.5$ in 3D case.

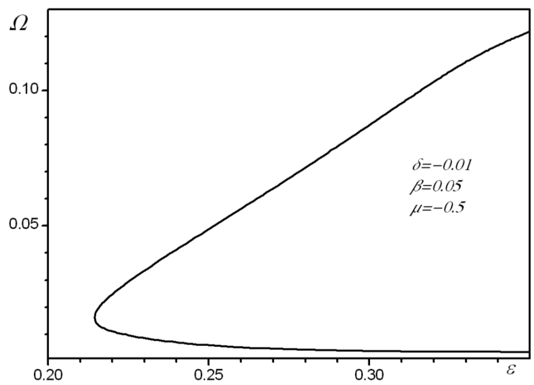


Fig. 4. Upper stable and lower unstable bifurcation curve of spatial frequency $\Omega$ as functions of the parameter $\varepsilon$ in 3D case.

an eigenvalue. For each set of dissipative parameters of a two-dimensional system it is necessary to adjust simultaneously the amplitude and the spatial frequency in order to find by shooting method the corresponding eigenvalue of $\Omega$. As a consequence, in two-dimensional system for instance, for dissipative parameters $\mu = -0.65$, $\beta = 0.05$, and $\delta = -0.01$, we obtain a bifurcation curve for the amplitude $A$ as function of the dissipation parameter $\varepsilon$ in Fig. 1. Figure 2 represents the plot of the spatial frequency $\Omega$ versus the same parameter $\varepsilon$.

The upper branch of these bifurcation curves corresponds to stable solutions of CQGLE, and the lower one to unstable. For three-dimensional case, for the chosen set of parameters $\mu = -0.5$, $\beta = 0.05$, and $\delta = -0.01$, bifurcation curves for amplitude and spatial frequency are given respectively in Figs. 3 and 4.

In conclusion, stationary CQGLE reduces into a set of three coupled ordinary differential equations. These equations can be solved numerically using the shooting method. Resulting bifurcation curves are in good agreement with the results obtained using variational approach and established stability criterions confirmed by numerical simulations [5, 6].


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