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Proceedings of the International School and Conference on Optics and  
Optical Materials, ISCOM07, Belgrade, Serbia, September 3–7, 2007

## Time Delay in Thin Slabs with Kerr-Type Nonlinearity

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In this paper we analyzed the following model: a thin slab with Kerr nonlinearity placed between two semi-infinite samples of linear and nonmagnetic materials. A general relation between the bidirectional group delay and the dwell time is derived for the thin slab. It is shown that the group delay is equal to the dwell time plus a self-interference delay. Particular attention is given to solving the Helmholtz equation for this case. Detailed and rigorous treatment revealed that the solutions of the Helmholtz equation are given via elliptic functions of the first kind. The boundary conditions at the interfaces are determined precisely. Finally, we provide an overall procedure for numerical calculation of the dwell times.

PACS numbers: 42.25.Bs, 42.65.Hw, 42.6.Tg, 03.65.Xp

### 1. Introduction

It is well known that the tunneling represents a typically quantum-mechanical phenomenon. Soon after the discovery of tunneling, Condon has raised the question of the speed of the tunneling process (in 1931) [1]. The papers published in the nineteen fifties [2–4] have provided analytical expressions for the time delays, suggesting those times to be very short but finite. Since then, the matter of defining various delay times and the interpretation of obtained expressions, has been in the focus of research with both theoretical and applied quantum mechanics, which is illustrated by the large number of review papers on this subject (e.g. [5–7]).

On the other hand, given the deep analogy between the Schrödinger equation and the Helmholtz equation, and the fact that the tunneling is present in

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propagation of electromagnetic waves through optically heterogeneous media, certain amount of attention has been devoted to the problem of finding delay times in these conditions, as well. In that respect, the following papers may be very helpful: a paper by Winful [8], and experimental work of Enders and Nimtz [9], Steinberg [10] and Spielmann et al. [11].

In this paper we analyze the delay times of electromagnetic waves in a thin slab with Kerr-type nonlinearity (e.g. [12]). The presence of this type of nonlinearity introduces considerable complications to the analytical expression, but on the other hand, it allows for manifestation of novel effects.

## 2. Theoretical considerations

In case of a high-intensity pulse, the material response, which we shall take to be nonmagnetic, becomes dependent on the electric field intensity. If the material may be considered isotropic, its relative permittivity,  $\varepsilon$ , may be written as

$$\varepsilon = \varepsilon_L + \alpha_{NL} |\mathbf{E}|^2, \quad (1)$$

with only the lowest order of nonlinearity taken into account. Let us consider a slab of thickness  $L$  made of such a material, put in vacuum and irradiated with a transverse electric (TE) wave. We shall label the axis perpendicular to the slab with  $x$ , let the electric field be pointed along the  $y$ -axis and assume that the propagation constant along the  $z$ -axis is  $\beta$ . Further, we assume that the time variation of all the fields is described with single angular frequency  $\omega$  and that the propagation constant of the TE plane wave incident on the slab is  $k_0$ .

The Helmholtz equation within the slab reads

$$d^2 E_y / dx^2 + (\kappa^2 + \alpha_{NL} k_0^2 |E_y|^2) E_y = 0, \quad \kappa^2 = \varepsilon_L k_0^2 - \beta^2, \quad 0 < x < L. \quad (2)$$

with  $\beta = k_0 \sin \theta$ , where  $\theta$  is the angle of incidence. The reflected,  $E_r$ , and transmitted,  $E_t$ , fields are given in terms of the reflection,  $R$ , and transmission coefficient,  $T$ , with

$$E_r = R E_0 \exp(-ik_0 x \cos \theta) \quad \text{and} \quad E_t = T E_0 \exp(ik_0 x \cos \theta), \quad (3)$$

where  $E_0$  is the complex amplitude of the incident wave. To find  $R$  and  $T$ , we have to solve (2), which we rewrite as

$$\tilde{E}_y'' + \left(1 + 2|\tilde{E}_y|^2\right) \tilde{E}_y = 0, \quad \tilde{E}_y = E_y / p, \quad p = \sqrt{2/\alpha_{NL} \kappa} / k_0, \quad (4)$$

here and in rest of the paper, the symbol  $'$  denotes differentiation with respect to  $\xi$ , where  $\xi = \kappa x$ . Introducing real numbers  $r$  and  $\varphi$  with

$$\tilde{E}_y = r \exp(i\varphi), \quad (5)$$

Eq. (4) may be separated in two equations with real quantities. From the imaginary part of (4) we obtain

$$\varphi' = \alpha / r^2, \quad (6)$$

and the real part of (4) yields

$$r'' - \alpha^2/r^3 + r + 2r^3 = 0, \quad \text{i.e.} \quad r'^2 + \alpha^2/r^2 + r^2 + r^4 = C, \quad (7)$$

with  $C$  being a constant across the slab which we will, together with  $\alpha$ , evaluate later. Let us consider the cubic equation

$$t^3 + t^2 - Ct + \alpha^2 = 0, \quad (8)$$

the roots of which we label  $t_1, t_2$  and  $t_3$ . It is easy to verify that introducing a new variable  $u$  with

$$r^2 = \tau_2 u^2 + t_1, \quad \tau_2 = t_2 - t_1, \quad (9)$$

into (7) enables us to recast it as

$$u'^2 + (1 - u^2)(\tau_3 - \tau_2 u^2) = 0, \quad \tau_3 = t_3 - t_1. \quad (10)$$

Now, it is straightforward to write down the solution for  $u$  in terms of the Jacobi elliptic function sn, namely

$$x = F(u, q) = \int_0^u \frac{d\nu}{\sqrt{(1 - \nu^2)(1 - q^2 \nu^2)}} \Rightarrow u = \text{sn}(x, q). \quad (11)$$

However, before doing so, we shall point out the right boundary conditions and determine on which domain should (10) be integrated. Firstly, we know that  $E_y$  and  $dE_y/dx$  are continuous in every point, since they correspond to the tangential components of the electric and magnetic field, respectively. Secondly, since the  $x$  component of the time-averaged Poynting vector is constant and nonzero along the structure (we assume that the dissipative losses can be neglected), we have that  $r$  is strictly positive, meaning that  $dr/dx$  is, also, continuous along the structure and the same holds for  $\varphi$  and  $d\varphi/dx$ . Therefore, we may evaluate  $\alpha$  and  $C$  at  $x = L^+$ , giving

$$\alpha = \frac{\alpha_{\text{NL}} |E_t|^2 k_0^4 \cos^4 \theta}{2\kappa^4} \quad \text{and} \quad C = \frac{p^2 \alpha^2}{|E_t|^2} + \frac{|E_t|^2}{p^2} + \frac{|E_t|^4}{p^4}. \quad (12)$$

Taking a closer look at our problem, it should be clear that the solution is easier obtained if we take that  $|E_t|$  is given instead of  $E_0$ , which we find at the end. Therefore, in integrating (10) we take that the lower limit is the point where the field is calculated,  $\xi$ , and for the upper limit we take  $\xi = \kappa L$ . Finally, we obtain the solution for  $|E_y|$ :

$$\begin{aligned} |E_y|^2 &= p^2 \left\{ t_1 + \tau_2 \text{sn}^2 \left[ \pm i\sqrt{\tau_3} \kappa(L - x) + F \left( \frac{|E_t|^2/p^2 - t_1}{\tau_2}, q \right), q \right] \right\}, \\ q &= \sqrt{\frac{\tau_2}{\tau_3}}. \end{aligned} \quad (13)$$

To keep it short, we now briefly explain how  $R$  and  $T$  are found. First, we assume that  $|E_t|$  is given and calculate  $\alpha, C, t_1, t_2$ , and  $t_3$ . Then, using (13), we obtain the solution for  $|E_y|$  and its derivative, which can also be expressed in analytic form using

$$d[\text{sn}(x, q)]/dx = \text{cn}(x, q)\text{dn}(x, q), \quad (14)$$

with cn and dn being, also, known Jacobi elliptic functions. Then, using the boundary conditions at  $x = 0$  and  $|R|^2 + |T|^2 = 1$ , we obtain  $|E_0|, |R|$ , and  $|T|$ .

Next, we numerically integrate (6) to find the phase difference across the slab and, finally, we find numerically the phases of  $R$  and  $T$ ,  $\phi_r$  and  $\phi_t$ , respectively.

Now, we proceed to use these results and calculate the delay times for the Kerr-type nonlinear slab. First, we state the expression for the overall electromagnetic energy,  $W$ , within the slab (obtained directly from the Poynting theorem):

$$W = \frac{S\varepsilon_0}{2} \int_0^L \varepsilon (|E_y|^2) |E_y|^2 dx - \frac{Sk_0 \cos \theta E_0^2}{2\omega^2 \mu_0} \text{Im}(R), \quad (15)$$

where  $S$  is the cross-section surface of the structure, perpendicular to the  $x$ -axis. To determine the delay times through the thin slab, and the way they are interrelated, we will apply the procedure previously used in the literature for the analogous problem of electrons tunneling through a potential barrier. Equation (2) can be modified to read

$$\frac{d^2 E_y(x)}{dx^2} + \kappa^2 E_y(x) = 0, \quad \kappa^2 \equiv \varepsilon(|E_y|^2) \frac{\omega^2}{c^2} - \beta^2. \quad (16)$$

We first differentiate Eq. (16) with respect to  $\omega$ , and subsequently multiply it by  $E_y^*$  to obtain our first equation. Similarly, the conjugation of Eq. (16), followed by multiplication by  $dE_y/d\omega$  provides another equation, from which the first one should be subtracted and the end result integrated along the slab (from  $0^+$  to  $L^-$ ), leading to

$$P = E_y^* \frac{d^2 E_y}{dx d\omega} - \frac{dE_y^*}{dx} \frac{dE_y}{d\omega} \Big|_{0^+}^{L^-} = -\frac{2\omega}{c^2} \int_0^L \varepsilon |E_y|^2 dx. \quad (17)$$

And now, the value of  $P$  can readily be taken from Ref. [13], Eq. (8), in the form

$$P = -2k \left( |T|^2 \frac{d\varphi_0}{d\omega} + |R|^2 \frac{d\varphi_r}{d\omega} + \frac{\text{Im}(R)}{k} \frac{dk}{d\omega} \right) E_0^2, \quad k = k_0 \cos \theta. \quad (18)$$

From Eqs. (17) and (18) we obtain an important auxiliary result

$$\int_0^L \varepsilon(|E_y|^2) |E_y|^2 dx = \frac{kc^2}{\omega} \left( \tau_g + \frac{\text{Im}(R)}{k} \frac{dk}{d\omega} \right) E_0^2 \sigma, \quad (19)$$

$$\sigma = \left\{ 1 - \frac{\sin^2 \theta + \omega \alpha_{\text{NL}} \text{Re} \left( E_y \frac{dE_y^*}{d\omega} \right)}{\varepsilon(|E_y|^2)} \right\}^{-1}$$

where we have used the definition of group delay from [7], i.e.  $\tau_g = |T|^2 \frac{d\varphi_0}{d\omega} + |R|^2 \frac{d\varphi_r}{d\omega}$  ( $\varphi_0 \equiv kL + \varphi_t$ ). With the use of (19) we arrive to the expression for the total energy within the slab

$$W = \frac{S\varepsilon_0 c^2 E_0^2}{2\omega} \left[ \sigma \tau_g + \text{Im}(R) \left( \frac{1}{k} \frac{dk}{d\omega} - \frac{1}{\omega} \right) \right]. \quad (20)$$

The input power is  $P_{\text{in}} = S\varepsilon_0 c^2 E_0^2 / 2\omega$ , so (20) reduces to

$$\frac{W}{P_{\text{in}}} = \tau_g \sigma + \text{Im}(R) \left( \frac{1}{k} \frac{dk}{d\omega} \sigma - \frac{1}{\omega} \right). \quad (21)$$

By defining the dwell time  $\tau_d$  as  $\tau_d = W/P_{\text{in}}$ , and the interface time as  $\tau_i = -\text{Im}(R) \left( \frac{1}{k} \frac{dk}{d\omega} - \frac{1}{\omega} \right) W/P_{\text{in}}$ , we obtain

$$\tau_d = \sigma \tau_g - \tau_i. \quad (22)$$

### 3. Numerical results

Given that the group delay requires the knowledge of derivatives  $d\varphi_r/d\omega$  and  $d\varphi_0/d\omega$ , it is convenient to start by determining the dwell time according to the formula

$$\tau_d = \frac{\omega}{kc^2 E_0^2} \left[ \int_0^L (\varepsilon_L |E_y|^2 + \alpha_{\text{NL}} |E_y|^4) dx \right] - \frac{\text{Im}(R)}{\omega}, \quad (23)$$

and then we calculate the interference time,  $\tau_i$ . Finally, the group delay is found according to  $\tau_g = (\tau_d + \tau_i)/\sigma$ .

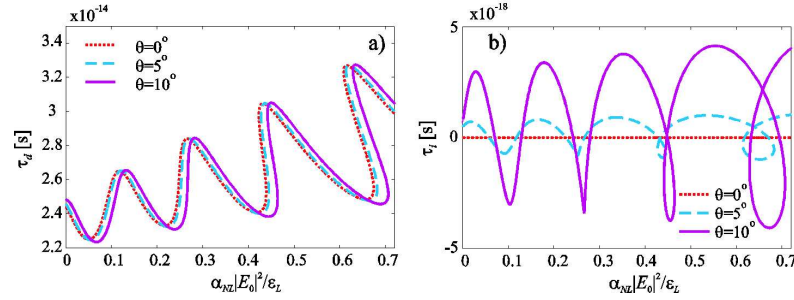


Fig. 1. Dependence of (a)  $\tau_d$  and (b)  $\tau_i$  on  $\alpha_{\text{NL}}|E_0|^2/\varepsilon_L$  drawn for several values of the incident angle  $\theta$ . Calculations were carried out for  $\lambda_0 = 1 \mu\text{m}$ ,  $L = 5 \mu\text{m}$ , and  $\varepsilon_L = 2$ .

Figure 1 shows the typical behaviour of delay times as the field intensity is increased. For larger nonlinearities several stable states exist to which correspond different delay times.

### 4. Conclusion

This paper provides a comprehensive analysis of the problem of calculating the delay times (dwell time, group time, and interference time) which characterize the transmission of electromagnetic waves through a thin slab with Kerr-type nonlinearity present. Particular consideration is given to the complex task of determining the field distribution within the slab. For this purpose, the Helmholtz equation is decomposed into two equations, one describing the amplitude of the field, and the other describing the phase of the field. While the second equation can easily be reduced to a simple integral equation, the solutions of the first one are given via elliptic functions. A simple analysis shows that all the required

constants can be obtained from two boundary conditions. Upon resolving the field distribution, in the second part of the paper, we derive the appropriate expressions for all three types of delay times and propose an adequate sequence in which they should be numerically evaluated. Based on the theoretical analysis presented here, our further work will focus on calculating the delay times in particular structures and the discussion of acquired numerical data.

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