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Dissipative Optical Solitons

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The generation and nonlinear dynamics of multi-dimensional optical dissipative solitonic pulses are examined. The variational method is extended to complex dissipative systems, in order to obtain steady state solutions of the one-, two-, and three-dimensional complex cubic-quintic Ginzburg–Landau equation. A stability criterion is established fixing a domain of dissipative parameters for stable steady state solutions. Following numerical simulations, evolution of even asymmetric input pulse from this domain leads to stable dissipative solitons and light bullets.

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1. Introduction

An isolated system tends always to the thermodynamical equilibrium. A nonlinear system open to the exchange of energy and mater with outside world can reach a steady state far from the thermodynamical equilibrium. Following Prigogine, such a system may self-organize into a dissipative structure [1]. Therefore, above a critical value of a control parameter ε the system abandons the unstable thermodynamical branch of the bifurcation diagram, becoming structured on the nonthermodynamical one, stabilized by permanent exchange of energy and matter with the external world. In opposition with always stable equilibrium structure, such a dissipative structure becomes stable when it reaches dynamic equilibrium. Complex nonlinear dissipative systems are now subject of very broad interest [2]. Wide class of such systems, ranging from nonlinear optics, plasma physics, and fluid dynamics to superfluidity, superconductivity, and Bose–Einstein condensates, can be modelled by complex Ginzburg–Landau equations [3]. Solitons belong to this class. The soliton is a temporal, spatial, or spatiotemporal localized structure conserving its shape after collision with another soliton. A soliton is generated due to the balance between linear and nonlinear effects. Being self-organized and self-maintained a soliton is astonishingly robust resisting to various perturbations. Taking into account these properties the soliton appears as a best candidate for transport and processing of information. Temporal solitons may soon become the principal carrier in telecommunication in dispersion compensated optical fibre transmission systems [4]. Spatiotemporal solitons can be used in all-optical signal processing since they are self-guided in bulk media carrying big power for a small dissipated energy. Stable operation of laser systems, closely related to the issue of dissipative soliton stability, is crucial for generating ultrashort pulses [5]. Spatiotemporal soliton completely localized in all transverse coordinates x, y, and time t is called light bullets when both diffraction and dispersion are compensated by spatiotemporal self-focusing [6]. A prerequisite to establish a bridge between the theory and the experiment is to consider dissipative systems. In such systems, the solitonic structure can be preserved if appropriate gains match linear and nonlinear losses. Optical solitons as a form preserving self-confined dissipative structures can be described by the multidimensional complex cubic-quintic Ginzburg–Landau equations (CQGLE) [7]. Such nonintegrable systems can be solved only numerically. However, an analytical approach, even though approximate, is needed in order to guide simulations and to avoid tedious numerical computations necessary to determine the stability domain point by point. In recent publication we used the variational method extended to dissipative systems, to establish this stability domain of parameters for CQGLE with radially symmetric input [8]. Indeed, an analytical stability criterion for dissipative one-, two-, and three-dimensional solitons is established and confirmed by exhaustive numerical simulations. Such a criterion provides analytically a broad domain of input parameters for generation of stable (3 + 1)-dimensional dissipative light bullets.

2. Ginzburg–Landau equation with radially symmetric input

In order to generate one- or few-parameters family of solitons with transverse dimension $D = 1, 2, 3$, the diffraction and/or dispersion have to be compensated by spatial and/or temporal self-focusing [6]. However, real systems are generally dissipative, thus, linear and nonlinear gain and loss have to be taken into account reducing the family of solitons for a given set of dissipative parameters into a fixed double solution [8]. Dynamics of dissipative solitons can be described by a $(D+1)$-dimensional nonlinear CQGLE for the normalized complex envelope of the optical field E:

$$\mathrm{i}\frac{\partial E}{\partial z} + \Delta E + |E|^2 E + \nu|E|^4 E = \mathrm{i}\delta E + \mathrm{i}\varepsilon|E|^2 E + \mathrm{i}\mu|E|^4 E + \mathrm{i}\beta\Delta E \equiv \mathbb{Q}, \quad (1)$$

where $\Delta E = r^{1-D}\partial(r^{D-1}\partial E/\partial r)/\partial r$ is the D-dimensional Laplacian describing beam diffraction and/or anomalous group velocity dispersion. In radially symmetric CQGLE second order derivatives are made with respect to the light bullet radius $r = \sqrt{x^2 + y^2 + t^2}$ imposing a constraint to independent transverse space (x and y) and time (t) variables [6]. In order to prevent the wave collapse the saturating nonlinearity is required. Therefore, cubic and quintic nonlinearity have to have opposite signs, i.e., parameter ν is negative. In order to have a stable pulse background, the linear dissipation term has to correspond to loss, the parameter δ must be always negative. The cubic and quintic gain-loss terms contain respectively parameters ε and μ. The last term accounts for the parabolic gain and must be positive $\beta > 0$. A prerequisite for generation of dissipative solitons is a simultaneous balance of not only diffraction and/or dispersion with self-focusing but also gain with loss. First, the variation approach has to be extended to complex dissipative systems described by CQGLE.

The total Lagrangian $\mathbb{L} = \mathbb{L}_{\mathrm{C}} + \mathbb{L}_{\mathrm{Q}}$ of the system described by Eq. (1) contains besides a conservative part $\mathbb{L}_{\mathrm{C}}$ also a dissipative part $\mathbb{L}_{\mathrm{Q}}$. Following Hamilton's principle the Lagrangian integral is stationary under condition that the Euler–Lagrange equation corresponds to Eq. (1). The trial function of Gaussian shape

$$E = A(z)\exp\left(-\frac{r^2}{2R(z)^2} + \mathrm{i}C(z)r^2 + \mathrm{i}\,\Psi(z)\right), \tag{2}$$

is expressed as a functional of amplitude A, pulse width R, wave front curvature C, and phase Ψ [8]. Optimisation of each of these functions gives one of four Euler–Lagrange equations averaged over transverse coordinates

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial L_{\mathrm{c}}}{\partial \eta'}\right) - \frac{\partial L_{\mathrm{c}}}{\partial \eta} = 2\Re\int r^{D-1}\mathrm{d}r\mathbb{Q}\frac{\partial E^*}{\partial \eta} \equiv Q_\eta, \tag{3}$$

where $\Re$ denotes the real part. The averaged conservative Lagrangian is denoted by $L_{\mathrm{c}} = \int \mathrm{d}r\mathbb{L}_{\mathrm{C}}$. In the dissipative case the power $P = A^2R^D$ is no longer a constant [6]. The parameter δ is always negative $\delta = -|\delta|$ [8]. It is renormalized as follows: $\delta_* = |\delta|R_*^2$ where $R_* = (8/3)^{1/2}(4/3)^{D/4}$. All remaining dissipative parameters are divided by δ_* and renormalized in order to be expressed in an unique form valid for different dimensions D: $\varepsilon_0 = 2\varepsilon/\delta_*$, $\mu_0 = 3\mu/2\delta_*$, and $\beta_0 = D\beta/2\delta_*$. All other quantities are also renormalized: , $R/R_* \to R$, $z/R_*^2 \to z$, $R_*^2C \to C$, and $A/A_* \to A$, where $A_* = (3/4)^{1/2}(3/2)^{D/4}$. Therefore, within variational approximation, to the partial differential CQGLE corresponds a set of four coupled first order differential equations (FODEs)

$$\frac{\mathrm{d}A}{\mathrm{d}z} = \left(\frac{4+D}{4}\varepsilon_0A^3 + \frac{3+D}{3}\mu_0A^5 - \frac{2\beta_0}{R^2}A - A\right)\delta_* - 2DCA \equiv F_A, \tag{4}$$

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \left(\frac{2\beta_0}{DR} - 4\beta_0R^3C^2 - \frac{\varepsilon_0}{2}RA^2 - \frac{2\mu_0}{3}RA^4\right)\delta_* + 4RC \equiv F_R, \tag{5}$$

$$\frac{\mathrm{d}C}{\mathrm{d}z} = \frac{1}{R^4} - \frac{A^2}{R^2} - \frac{\nu A^4}{R^2} - 4C^2 - 8\frac{\beta_0\delta_*}{DR^2}C \equiv F_C, \tag{6}$$

and

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = 4\beta_0\delta_*C - \frac{D}{R^2} + \frac{4+D}{2}A^2 + \frac{3+D}{2}\nu A^4 \equiv \Omega. \tag{7}$$

The steady state solutions can be obtained from Eqs. (4)–(6) for vanishing derivatives of amplitude, width, and curvature. These variables are expanded up to the small parameter $\theta = \max\{\beta,\delta,\varepsilon,\mu\} \ll 1$; $R = R_0 + \mathcal{O}(\theta^2)$ and $A = A_0 + \mathcal{O}(\theta^2)$, as well as $C = C_1\delta_* + \mathcal{O}(\theta^2)$. The lowest order width $R = A^{-1}(1+\nu A^2)^{-\frac{1}{2}}$ and the propagation constant $\Omega = 0.5A^2[(4-D) + \nu(3-D)A^2]$ depend only on the amplitude as in the conservative case [6]. Variationally obtained families of conservative steady state solutions for $D = 1$, 2, and 3 reduces, in the dissipative case, to a fixed double solution for a given set of dissipative parameters. Indeed, the amplitude as a steady state solution of Eqs. (4)–(6), has two discrete values A^+ and A^-:

$$A^{\pm} = \sqrt{\frac{(\beta_0 - \varepsilon_0) \pm \sqrt{(\beta_0 - \varepsilon_0)^2 + 4(\mu_0 - \nu\beta_0)}}{2(\mu_0 - \nu\beta_0)}}. \tag{8}$$

The existence of either unique solution A^+ or double solution ($A^- > A^+$) implies a cubic gain $\varepsilon > 0$ and a quintic loss $\mu < 0$.

Unique solutions are separated in Fig. 1 from double solutions by the a-line corresponding to $\mu_0 = 1 - \varepsilon_0 + \beta_0(1 + \nu)$. The domain of double solutions is also limited by the d-parabola expressed as $(\beta_0 - \varepsilon_0)^2 + 4(\mu_0 - \nu\beta_0) = 0$. Double solution for the set of dissipative parameters $\varepsilon_0 = 19$, $\mu_0 = -23.5$, $\beta_0 = 1.5$, $\delta_* = 0.001$, and $\nu = -1$ is illustrated by a diamond superposed on a triangle in Fig. 1.

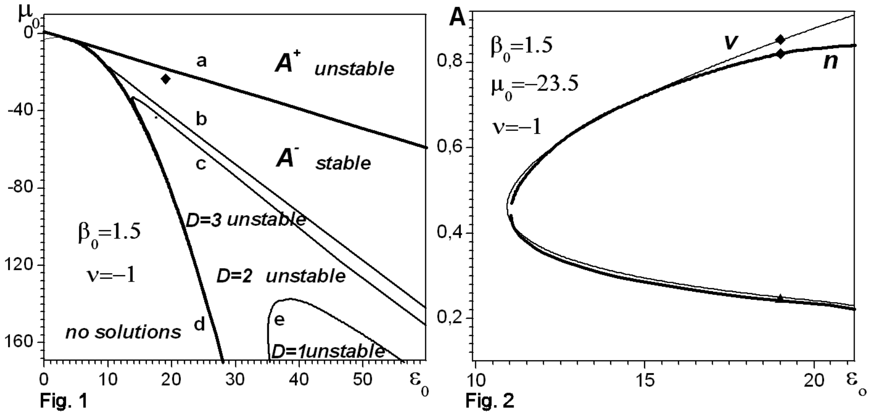


Fig. 1. Domain of stable solutions A^-.

Fig. 2. Upper stable and lower unstable branches of variational (v) and numerical (n) bifurcation curves.

Another striking difference with conservative systems is the nonzero wave front curvature $C = A^2[\varepsilon_0/8 - \beta_0/2D + (\mu_0/6 - \nu\beta_0/2D)A^2]\delta_*$ [6]. The gain–loss balance together with the compensation of diffraction and/or dispersion with saturating nonlinearity can be realized only for nonzero curvature fixed steady state solutions. Only stable solutions can be solitons. Variationally obtained Euler–Lagrange equations are the starting point in order to establish a stability criterion using the method of Lyapunov's exponents [1]. A Jacoby determinant is constructed from derivatives with respect to amplitude, width, and curvature of terms F_A, F_R, and F_C of Eqs. (4)–(6) taken in steady state. Following Lyapunov, steady state solutions of the set of nonlinear FODEs are stable if and only if the real part of solutions λ of cubic equation $\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0$ are negative [1]. In order to have Lyapunov's stability, Hurwitz's conditions must be fulfilled: coefficients α_3 and α_2 as well as their combination $\alpha_6 = \alpha_1\alpha_2 - \alpha_3$ have to be positive. Therefore, the stability criterion for variationally obtained steady state solutions of D-dimensional CQGLE, up to θ, is explicitly expressed

$$\alpha_2 = 4A^4(1+\nu A^2)[2-D-2\nu(D-1)A^2] + \mathcal{O}(\theta^2) > 0,$$

$$\alpha_3 = 16A^4(1+\nu A^2)[(\nu\varepsilon_0-\mu_0)A^4-2\nu A^2-1]\delta_* + \mathcal{O}(\theta^3) > 0$$

with

$$\alpha_1 = \left[-8/D + \varepsilon_0(8/D-1-D/2)A^2 + \mu_0(8/D-8/3-4D/3)A^4\right]\delta_*.$$

The coefficient α_3 is everywhere positive on the solution A^- and negative on A^+. As a consequence, only solutions A^- can be potentially stable. Above tilted horseshoe b, c, and e corresponding to $\alpha_4 = 0$ the solution A^- of appropriate dimension is stable. This solution is a stable focus since $\alpha_5 = 4\alpha_2^3 > 0$. For instance, for the set of dissipative parameters chosen above, the triangle on the lower unstable branch and the diamond on the upper stable branch of the v-curve in Fig. 2 representing the amplitude as a function of a dissipative parameter ε_0, correspond respectively to A^+ and to A^-. Therefore, the stability criterion implies that any steady state solution of 1-, 2-, or 3-dimensional CQGLE belonging to the established stable domain of dissipative parameters will be stable. This criterion is tested using numerical simulations of CQGLE. Input pulse chosen in the stable domain of parameters is not yet a stable soliton since the variationally obtained v-bifurcation curve does not coincide with the exact numerically obtained n-curve in Fig. 2 (see two diamonds). However, following our numerical simulations, the input pulse with parameters from the established stable domain evolves towards the stable dissipative soliton on the n-bifurcation curve. Therefore, whenever an input pulse belongs to the stable domain the final stage of evolution is always a stable dissipative soliton. Therefore, generated Prigogine's dissipative structure which is self-maintained against dissipation, is a stable dissipative soliton.

3. Ginzburg–Landau equation with asymmetric input

In order to take into account experimental conditions, the light bullet has to be generated starting from an input pulse asymmetric with respect to transverse coordinates x, y, and time t [9]. Therefore, we extend the synergy of our analytical and numerical approach in order to study a much broader class of Ginzburg–Landau systems involving asymmetric input pulses. As a consequence, here we study (3 + 1)-dimensional CQGLE describing separately diffractions following x and y coordinates and anomalous group velocity dispersion in time t without any constraint, i.e. in Eq. (1) Laplacian is $\Delta E = \partial^2E/\partial x^2 + \partial^2E/\partial y^2 + \partial^2E/\partial t^2$. For a given set of parameters the continuous family of solutions reduces to a fixed one representing an isolated attractor [10]. The independent treatment of all three transverse coordinates involves an asymmetric trial function

$$E = A\exp\left(-\frac{x^2}{2X^2} - \frac{y^2}{2Y^2} - \frac{t^2}{2T^2} + \mathrm{i}Cx^2 + \mathrm{i}Sy^2 + \mathrm{i}Gt^2 + \mathrm{i}\,\Psi\right) \qquad (9)$$

as functional of amplitude A, temporal (T) and spatial (X and Y) pulse widths, anisotropic wave front curvatures C and S, chirp G, and phase Ψ.

Within variational approximation, to the partial differential CQGLE corresponds now a set of eight coupled FODEs resulting from the variations in amplitude

$$\frac{\mathrm{d}A}{\mathrm{d}z} = \delta A + \frac{7\varepsilon A^3}{8\sqrt{2}} + \frac{2\mu A^5}{3\sqrt{3}} - \frac{\beta A}{X^2} - \frac{\beta A}{Y^2} - \frac{\beta A}{T^2} - 2(C+S+G)A, \tag{10}$$

different widths X, Y, and T,

$$\frac{\mathrm{d}X}{\mathrm{d}z} = 4CX - \frac{\varepsilon X A^2}{4\sqrt{2}} - \frac{2\mu X A^4}{9\sqrt{3}} + \frac{\beta}{X} - 4\beta C^2 X^3, \tag{11}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = 4SY - \frac{\varepsilon Y A^2}{4\sqrt{2}} - \frac{2\mu Y A^4}{9\sqrt{3}} + \frac{\beta}{Y} - 4\beta S^2 Y^3, \tag{12}$$

$$\frac{\mathrm{d}T}{\mathrm{d}z} = 4GT - \frac{\varepsilon T A^2}{4\sqrt{2}} - \frac{2\mu T A^4}{9\sqrt{3}} + \frac{\beta}{T} - 4\beta G^2 T^3, \tag{13}$$

different wave front curvatures C, S, and G,

$$\frac{\mathrm{d}C}{\mathrm{d}z} = \frac{1}{X^4} - \frac{A^2}{4\sqrt{2}X^2} + \frac{2\mu A^4}{9\sqrt{3}X^2} - 4C^2 - \frac{4\beta C}{X^2}, \tag{14}$$

$$\frac{\mathrm{d}S}{\mathrm{d}z} = \frac{1}{Y^4} - \frac{A^2}{4\sqrt{2}Y^2} + \frac{2\mu A^4}{9\sqrt{3}Y^2} - 4S^2 - \frac{4\beta S}{Y^2}, \tag{15}$$

$$\frac{\mathrm{d}G}{\mathrm{d}z} = \frac{1}{T^4} - \frac{A^2}{4\sqrt{2}T^2} + \frac{2\mu A^4}{9\sqrt{3}T^2} - 4G^2 - \frac{4\beta G}{T^2}, \tag{16}$$

and phase

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = 2\beta(C+S+G) - \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{T^2} + \frac{7A^2}{8\sqrt{2}} - \frac{2\nu A^4}{3\sqrt{3}}. \tag{17}$$

The exact steady state solutions are obtained from Eqs. (10)–(16) for zero z derivatives of amplitude, widths, and curvatures. The only possible steady state solutions are symmetric with equal widths $X = Y = T$ and curvatures $C = S = G$. The steady state solutions of seven coupled FODEs are stable if and only if following Hurwitz's conditions the real part of solutions λ of equation

$$(\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3)(\lambda^2 + \alpha_4\lambda + \alpha_5)^2 = 0$$

are non-positive. The stability criterion for steady state solutions of Ginzburg–Landau equation is explicitly expressed up to δ_0 as

$$\alpha_2 = 0.07A^4(1.38 - \nu A^2)(4\nu A^2 - 1.38) + \mathcal{O}(\theta^2) > 0,$$

$$\alpha_3 = 0.02A^4(1.38 - \nu A^2)^2[(4\varepsilon - 3\beta)A^2 - 22.63|\delta|] + \mathcal{O}(\theta^3) > 0,$$

$$\alpha_4 = [0.35(\varepsilon + 2\beta)A^2 + 0.29(\mu - 2\nu\beta)A^4] + \mathcal{O}(\theta^3) > 0,$$

and

$$\alpha_5 = \mathcal{O}(\theta^2)$$

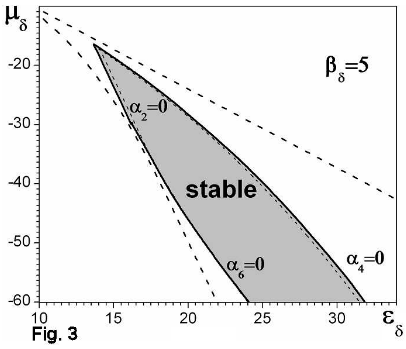
as well as

$$\alpha_6 = \alpha_1\alpha_2 - \alpha_3$$

with

$$\alpha_1 = (0.06\varepsilon A^2 - 0.77\mu A^4 - 2.67|\delta|) + \mathcal{O}(\theta^3).$$

As a consequence, in the (ε, μ)-domain in Fig. 3 only A^- solution is stable in the shaded region between curves $\alpha_2 = 0$ and $\alpha_6 = 0$ (separated by a square), as well as $\alpha_4 = 0$. Full curves correspond to the exact solution of the same set of equations solved parametrically. Input pulse chosen in the stable domain of parameters is not yet a light bullet since the variationally obtained bifurcation curve v in Fig. 4 corresponding to the power P as a function of the parameter $\varepsilon_\delta = \varepsilon/|\delta|$ is only a good approximation of exact bifurcation curve n obtained by numerical solution of Eq. (1).



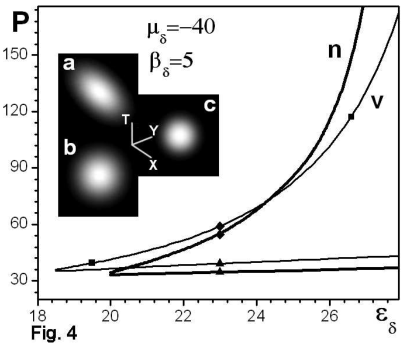


Fig. 3. Stability domain of A^- solutions computed exactly (full curves) and up to θ (dashed curves). $\varepsilon_\delta = \varepsilon/|\delta|$, $\mu_\delta = \mu/|\delta|$, $\beta_\delta = \beta/|\delta|$.

Fig. 4. Stable and unstable branches of variational (v) and numerical (n) curves with insets corresponding to asymmetric (a) and symmetric (b) inputs as well as to soliton (c).

The analytically predicted domain of stability is exhaustively checked point by point using numerical simulations of Eq. (9); a stable soliton is generated from each point. If the stable steady state solution is taken as the input in numerical simulations (inset (b) in Fig. 4), it will evolve shrinking towards the stable dissipative soliton (inset (c)) represented by the diamond on the numerical simulations (inset (b) in Fig. 4), it will evolve shrinking towards the stable dissipative soliton (inset (c)) represented by the diamond on the exact n-curve. However, the same final soliton can be obtained starting from an asymmetric, i.e., ellipsoidal input pulse (inset (a)) with the same set of dissipative parameters belonging to the stable domain, as numerical simulations demonstrate.

4. Conclusion

In conclusion, in order to obtain steady state solutions of the $(D + 1)$--dimensional CQGLE, an analytical approach is developed based on the extension of the variational method to dissipative systems. In order to treat simultaneously all three dimensions, the D-dimensional Laplacian in CQGLE has to be centrosymmetric excluding asymmetric input pulses. Based on this variational approach and the method of Lyapunov exponents, a general stability criterion for dissipative D-dimensional solitons is established. Input pulses generated in the proposed domain of dissipative parameters, evolve towards stable dissipative solitons, as numerical simulations of CQGLE confirm. CQGLE is also treated for the non-spherically symmetric input, using jointly numerical and analytical approach. A new stability criterion based on FODEs without spherical symmetry, is established in order to select stable steady state solutions from the domain of dissipative parameters obtained analytically through exact parametric resolution. Stability of the analytically predicted domain is point by point confirmed using numerical simulations. Following our numerical simulations each asymmetric input pulse with dissipative parameters chosen from the stability domain determined by that criterion, will evolve attracted by the fixed exact solution in order to self-organize into a dissipative light bullet. It is worthwhile to stress that even very asymmetric input pulses (like in Fig. 4), which are far from stable spherically symmetric steady states, but for the same dissipative parameters, always self-organize into soliton. The opportunity to treat analytically and numerically asymmetrical input pulses propagating toward stable and robust dissipative light bullets, opens possibilities for diverse practical applications for conception of all-optical transmission systems, signal processing, and mode-locked laser generating ultrashort pulses.

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